PMM U.S.S.R.,Vol.44,pp.350-356 0021-8928/81/3 0350 $7.50/0

UDC 539.3

ON UNIQUENESS CONDITIONS IN THE SMALL FOR THE STATE OF HYDROSTATIC COMPRESSION OF A SOLID*

L. M. ZUBOV

The governing inequality assuring the stability of any homogeneous anisotropic elastic body subjected to a multilateral follower pressure is formulated. A comparison is given between the uniqueness conditions for the state of hydrostatic compression and the governing inequalities of Hill, Hadamard, and Coleman-Noll. The variant when part of the body boundary slides along a smooth solid surface is also considered for an isotropic incompressible body. If the external pressure is a dead load, the condition mentioned does not assure uniqueness of the solutions of the neutral equilibrium equations. The problem of determining the critical pressure for dead loading of an isotropic incompressible body reduces to a problem of minimizing a functional.

The condition of no adjacent equilibrium modes (i.e., stability in the sense of a static Euler criterion) for an elastic body loaded by multilateral hydrostatic pressure was proposed by Lur'e /1/ as the necessary condition imposing a constraint on the mode of the law of the material state. Constraints on the dependence of the specific potential energy of an isotropic material on the relative change in volume were found from this condition in /1/.

1. The equilibrium equations for an elastic body linearized in the neighborhood of a certain initial state of stress have the following form in the absence of additional mass forces

$$\nabla\cdot\Theta = 0,\quad \Theta = \mathrm{T}^{\cdot} + \mathrm{T}\,\mathrm{tr}\,\mathrm{L} - \mathrm{L}^T\cdot\mathrm{T} \tag{1.1}$$

$$\mathrm{L} = \nabla\mathbf{w},\quad \mathbf{w} = \mathbf{R}^{\cdot} = \left[\frac{d}{d\eta}(\mathbf{R}+\eta\mathbf{w})\right]_{\eta=0},\quad \mathrm{T}^{\cdot} = \left[\frac{d}{d\eta}\mathrm{T}(\mathbf{R}+\eta\mathbf{w})\right]_{\eta=0}$$

Here ∇ is the nabla-operator in the metric of the initial state of strain $\mathbf{R}$ is a radius vector, T is the Cauchy stress tensor, and $\mathbf{w}$ is the vector of small additional displacements. The upper dot denotes linear increments of the tensor, vector, or scalar quantities in a fixed material particle which are due to the superposition of additional displacements. These increments can be interpreted as material rates of change in the appropriate quantities if the parameter η is identified with time and the vector $\mathbf{w}$ with the particle velocity vector.

For an elastic material the Cauchy stress tensor is a function of the gradient of the strain C. Taking into account that $\mathrm{C}^{\cdot} = \mathrm{C}\cdot\mathrm{L}$, we can write

$$\mathrm{T}^{\cdot} = \left[\frac{d}{d\eta}\mathrm{T}(\mathrm{C}+\eta\mathrm{C}\cdot\mathrm{L})\right]_{\eta=0} \tag{1.2}$$

It follows from (1.1) and (1.2) that the tensor Θ is a linear function of the tensor L. Moreover, it depends nonlinearly on the gradient of the strain corresponding to the initial strain configuration. We rewrite the representation of the tensor Θ as follows:

$$\Theta(\mathrm{C},\mathrm{L}) = \mathrm{S} - \mathrm{T}\cdot\Omega + \Omega\cdot\mathrm{T} + \mathrm{T}\,\mathrm{tr}\,\mathrm{L} - \mathrm{L}^T\cdot\mathrm{T}$$
$$\mathrm{S} = \mathrm{T}^{\cdot} + \mathrm{T}\cdot\Omega - \Omega\cdot\mathrm{T},\quad \Omega = {}^1/_2(\mathrm{L}^T - \mathrm{L}) \tag{1.3}$$

The symmetric tensor S is the rate of change of the stress tensor in the Jaumann sense /2,3/.

The law of the state of an ideal elastic (hyperelastic) material has the form /1/

$$\mathrm{T} = \mathrm{C}^T\cdot\mathrm{P}\cdot\mathrm{C},\quad \mathrm{P} = 2J^{-1}dW/d\mathrm{G},\quad \mathrm{G} = \mathrm{C}\cdot\mathrm{C}^T,\quad J = \det\mathrm{C} \tag{1.4}$$

Here G is a measure of the Cauchy strain, and $W(\mathrm{G})$ is the density of the potential strain energy per unit volume of the measured configuration. From (1.4) we have

$$\mathrm{S} = \mathrm{C}^T\cdot\mathrm{P}^{\cdot}\cdot\mathrm{C} + \varepsilon\cdot\mathrm{T} + \mathrm{T}\cdot\varepsilon,\quad \varepsilon = {}^1/_2(\mathrm{L}+\mathrm{L}^T) \tag{1.5}$$

Since $\mathrm{G}^{\cdot} = 2\mathrm{C}\cdot\varepsilon\cdot\mathrm{C}^T$, it follows from (1.4) and (1.5) that the tensor S is independent of the spin Ω and is a linear function of the strain rate tensor ε

$$\mathrm{S}(\mathrm{C},\varepsilon) = \mathrm{K}(\mathrm{C})\cdot\cdot\,\varepsilon \tag{1.6}$$

By using (1.4) it can be shown that the fourth-rank tensor K allows of the following

*Prikl.Matem.Mekhan.,44,No.3,497-506,1980 representation (E is the unit tensor)

$$\mathrm{K}(\mathrm{C}) = \Phi(\mathrm{C}) - \mathrm{TE} \tag{1.7}$$

where the tensor Φ is symmetric relative to permutation of pairs of subscripts $\Phi_{mnij} = \Phi_{ijmn}$.

We call the state of an elastic body loaded by a multilaterial uniform hydrostatic pressure of intensity p a p-configuration. Such a loading evokes a certain affine deformation in a homogeneous body, whose gradient will be denoted by C_p. We shall also denote the measures of the deformation corresponding to the p-configuration by the subscript p. Since $\mathrm{T} = -p\mathrm{E}$ in a p-configuration, we obtain from (1.3)

$$\Theta = \mathrm{S}(\mathrm{C}_p, \varepsilon) + p(\nabla \mathbf{w}^T - \mathrm{E}\nabla\cdot\mathbf{w}) \tag{1.8}$$

The boundary conditions on the surface of a body O in a problem on the superposition of a small, on a finite deformation with the following nature of the pressure taken into account, have the form /1/

$$\mathbf{N}\cdot\Theta = p\mathbf{N}\cdot(\nabla \mathbf{w}^T - \mathrm{E}\nabla\cdot\mathbf{w}) \tag{1.9}$$

Here $\mathbf{N}$ is the unit vector of the external normal to the surface O. Taking account of the identity $\nabla\cdot(\nabla \mathbf{w}^T - \mathrm{E}\nabla\cdot\mathbf{w}) = 0$, and the representation (1.8), we write the boundary condition about small deformations from the p-configuration in the form

$$\nabla\cdot\mathrm{S} = 0 \text{ in volume } V, \ \mathbf{N}\cdot\mathrm{S} = 0 \text{ on } O \tag{1.10}$$

According to (1.7), the tensor of the "instantaneous elastic moduli" K does not generally possess the property of invariance relative to permutation of pairs of subscripts. However, this property is evidently satisfied in the p-configuration.

The boundary value problem (1.10) is completely identical to the problem for a homogeneous anisotropic body in classical linear elasticity theory /1/. Its solution is unique (to the accuracy of a rigid displacement) under the condition

$$\mathrm{S}(\mathrm{C}_p, \varepsilon)\cdot\cdot\,\varepsilon > 0 \quad \text{or} \quad \varepsilon\cdot\cdot\,\mathrm{K}(\mathrm{C}_p)\cdot\cdot\,\varepsilon > 0, \quad \forall\varepsilon \neq 0 \tag{1.11}$$

Let us note that the uniqueness of the solution of problem (1.10) will hold also in case the following is taken in place of condition (1.11):

$$\mathrm{S}(\mathrm{C}_p, \varepsilon)\cdot\cdot\,\varepsilon < 0, \quad \forall\varepsilon \neq 0 \tag{1.12}$$

However, in contrast to (1.12) the inequality (1.11) not only assures the uniqueness of the solution but also is related, in a definite sense, to the stability of the p-configuration. Indeed, the potential energy of a hyperelastic body, loaded over the whole surface by uniform hydrostatic pressure, is given by the expression /1/ (v is the body volume in the reference configuration)

$$\Pi = \iiint_v (W + pJ)\, dv \tag{1.13}$$

Referring to (1.3), formula (11) in /5/, and the relationship

$$\left[\frac{d^2}{d\eta^2}\det(\mathrm{C} + \eta\mathrm{C}\cdot\mathrm{L})\right]_{\eta=0} = J(\mathrm{tr}^2\,\mathrm{L} - \mathrm{L}\cdot\cdot\,\mathrm{L})$$

we arrive at the following expression for the second variation of the functional (1.13) in the neighborhood of the p-configuration:

$$\left[\frac{d^2}{d\eta^2}\Pi(\mathbf{R} + \eta\mathbf{w})\right]_{\eta=0} = \iiint_V \mathrm{S}(\mathrm{C}_p, \varepsilon)\cdot\cdot\,\varepsilon\, dV$$

2. Condition (1.11) which assures the uniqueness and stability of any p-configuration imposes a constraint on the response function of an elastic material, i.e., is a governing inequality. It can be extended (not by a unique method it is understood) to the whole configuration of an elastic body. The simplest and most natural extension of condition (1.11) will be the inequality

$$\mathrm{S}(\mathrm{C}, \varepsilon)\cdot\cdot\,\varepsilon > 0, \quad \forall\varepsilon \neq 0 \tag{2.1}$$

where C is any nonsingular tensor, i.e., the gradient of the strain corresponding to an arbitrary and not only a p-configuration. Since the strain rate tensor and the Jaumann derivative of the Cauchy stress tensor are tensors independent of the reference system, condition (2.1) does not contradict the requirement of invariance relative to the choice of the observer.

In the case of an isotropic material, we take some undistorted state /6/, for instance, a natural unstressed state, as the reference configuration. Then by using (1.5) it can be shown that the tensor S will be an isotropic function of the tensors ε and $\mathrm{g} = (\mathrm{C}^T\cdot\mathrm{C})^{-1}$, where g is a measure of the Almansi strain. The Cauchy stress tensor is an isotropic function of the tensor g in this case. Assuming a dependence between T and g to be mutually single-valued, it can be assumed that $S = S(T, \varepsilon)$. In such form the inequality (2.1) can be used for plastic materials also.

The law of the state of an isotropic elastic material can be given by an isotropic function σ /1/

$$\Sigma = \sigma(H), \quad \Sigma = A \cdot T \cdot A^T, \quad H = {}^1/_2 \ln G, \quad A = U^{-1} \cdot C, \quad U = G^{1/2} \tag{2.2}$$

Here A is the rotation tensor, U is the positive-definite square root of the Cauchy strain measure, and H is the logarithmic strain measure (the Hencky tensor). For a hyperelastic body we have /1/

$$J\Sigma = dW / dH, \quad \ln J = \mathrm{tr}\, H \tag{2.3}$$

Let us prove that the relationship

$$S(C_p, \varepsilon) \cdot\cdot\, \varepsilon = \Sigma^{\cdot}(H_p, H^{\cdot}) \cdot\cdot\, H^{\cdot} \tag{2.4}$$

is valid in any p-configuration of an isotropic body.

We evidently have for the p-configuration of an isotropic material

$$C_p = \alpha Q, \quad \alpha = (\rho / \rho_0)^{-1/3} > 0, \quad A_p = Q, \quad U_p = \alpha E, \quad H_p = \ln \alpha E \tag{2.5}$$

Here Q is an intrinsically orthogonal tensor, ρ is the material density in the p-configuration, and ρ_0 the density in the reference configuration. On the basis of (2.2) and (2.5), we obtain

$$H = {}^1/_2 \ln [(C_p + \eta C_p \cdot L) \cdot (C_p^T + \eta L^T \cdot C_p^T)] = \ln \alpha E + {}^1/_2 \ln (E + 2\eta Q \cdot \varepsilon \cdot Q^T) + O(\eta^2)$$

It hence follows that the formula

$$H^{\cdot} = Q \cdot \varepsilon \cdot Q^T \tag{2.6}$$

is valid in the p-configuration.

The rate of change of the rotation tensor is associated with the spin as follows /7/:

$$A^{\cdot T} \cdot A = \Omega + {}^1/_2 (j_1 j_2 - j_3)^{-1} [j_1^2 (g^{1/2} \cdot \varepsilon - \varepsilon \cdot g^{1/2}) + j_1 (\varepsilon \cdot g - g \cdot \varepsilon) + g \cdot \varepsilon \cdot g^{1/2} - g^{1/2} \cdot \varepsilon \cdot g] \tag{2.7}$$

Here j_1, j_2, j_3 are the principal invariants of the tensor $g^{1/2}$. From (2.2), (2.5) and (2.7) we obtain for the p-configuration

$$\Sigma^{\cdot} = Q \cdot (T^{\cdot} - \Omega \cdot T + T \cdot \Omega) \cdot Q^T \tag{2.8}$$

Here (2.4) results from (2.6) and (2.8).

The relationship (2.4) shows that the governing inequality

$$\Sigma^{\cdot}(H, H^{\cdot}) \cdot\cdot\, H^{\cdot} > 0, \quad \forall H^{\cdot} \neq 0 \tag{2.9}$$

is the sufficient condition for uniqueness in the small for the state of hydrostatic compression of an isotropic elastic body. The inequalities (2.1) and (2.9) are distinct for an arbitrary configuration but coincide, as shown above, for any p-configuration of an isotropic material.

The inequality (2.9) differs somewhat from the condition proposed in /8/ for the convexity of the specific energy W as a function of the logarithmic measure of the strain. By virtue of (2.3), this latter condition is written thus:

$$(J\Sigma)^{\cdot} \cdot\cdot\, H^{\cdot} > 0, \quad \forall H^{\cdot} \neq 0 \tag{2.10}$$

In a p-configuration the condition (2.10) becomes

$$S \cdot\cdot\, \varepsilon - p\, \mathrm{tr}^2\, \varepsilon > 0, \quad \forall \varepsilon \neq 0 \tag{2.11}$$

For a positive values of p, i.e., for a compressive pressure, (1.11) evidently follows from (2.11). Therefore, the condition of convexity of the specific energy in the Hencky strain tensor is sufficient for the uniqueness and stability of the state of hydrostatic compression of an isotropic body.

Let us note that condition (2.9) assures the stability of the p-configuration independently of the sign of p. It is impossible to say this about condition (2.10).

Let us compare (1.11) with the governing inequalities of Coleman-Noll and Hadamard /6/ obtained by extension. The former is written thus

$$\Theta(C, L) \cdot\cdot\, L^T > 0, \qquad \forall L = L^T = \varepsilon \neq 0 \tag{2.12}$$

while the Hadamard inequality has the form

$$\Theta(C, L) \cdot\cdot\, L^T \geqslant 0 \tag{2.13}$$

for all tensors L representable as the dyadic product of vectors $L = 2\mathbf{ab}$. The strengthened

Hadamard inequality

$$\Theta(\mathrm{C},\ 2\mathbf{ab})\cdot\cdot\mathbf{ba} > 0, \quad \forall\mathbf{a} \neq 0, \quad \mathbf{b} \neq 0 \tag{2.14}$$

is the condition /6/ of strong ellipticity of the differential equation system of elasticity theory.

According to (1.3), condition (2.12) becomes in the p-configuration

$$\mathrm{S}(\mathrm{C}_p,\ \varepsilon)\cdot\cdot\varepsilon - p\,(\mathrm{tr}^2\varepsilon - \mathrm{tr}\,\varepsilon^2) > 0, \quad \forall\varepsilon \neq 0 \tag{2.15}$$

Comparing (2.15) with (1.11), we see that the Coleman-Noll inequality does not assure uniqueness (to the accuracy of rigid displacement) of the boundary value problem (1.10).

Let us note that in the case of an isotropic hyperelastic material the inequality (2.15) results from the condition of convexity of the specific energy W as a function of the tensor $\mathrm{U} - \mathrm{E}$. This is proved by a calculation analgous to that performed in deriving (2.4).

Starting from (1.3), it can be verified that the strengthened Hadamard inequality will be the following in application to the p-configuration:

$$\mathrm{S}(\mathrm{C}_p,\ \mathbf{ab} + \mathbf{ba})\cdot\cdot(\mathbf{ab} + \mathbf{ba}) > 0, \quad \forall\mathbf{a} \neq 0, \quad \mathbf{b} \neq 0 \tag{2.16}$$

Evidently (2.16) follows from (1.11). The reverse assertion is not valid.

Indeed, let us examine the case of an isotropic material as illustration. According to (2.5), the tensor S will be a linear isotropic function of ε whose general representation has the form

$$\mathrm{S} = 2\mu(\alpha)\varepsilon + \lambda(\alpha)\mathrm{E}\,\mathrm{tr}\,\varepsilon \tag{2.17}$$

As is known from linear elasticity theory /1/, in application to (2.17), condition (1.11) is equivalent to the inequalities

$$\mu > 0, \quad 2\mu + 3\lambda > 0 \tag{2.18}$$

In application to (2.17), the inequality (2.16) yields

$$\mu a^2 b^2 + (\mu + \lambda)(\mathbf{a}\cdot\mathbf{b})^2 > 0 \tag{2.19}$$

The first, but not the second, inequality in (2.18) results from (2.19). In fact, since $(\mathbf{a}\cdot\mathbf{b})^2 \leqslant a^2b^2$ for $\lambda = -{}^3/_2\mu$, condition (2.19) is satisfied while the second inequality in (2.18) is not.

3. Only isochoric strains for which $J = 1$ are allowable in an incompressible elastic material, and the Cauchy stress tensor is determined by the strains of the particle neighborhoods to the accuracy of an arbitrary global tensor. In this case the representation of the tensor S has the form

$$\mathrm{S} = \mathrm{M}(\mathrm{C})\cdot\cdot\varepsilon + q\mathrm{E} \tag{3.1}$$

where q is an unknown function of the coordinates, and the linearized condition for incompressibility

$$J^{\cdot} = \mathrm{tr}\,\mathrm{L} = \mathrm{tr}\,\varepsilon = \nabla\cdot\mathbf{w} = 0 \tag{3.2}$$

is the additional equation to find it.

The solution of the boundary value problem (1.10) for an incompressible body is unique (if the rigid displacement is discarded) under the condition

$$\mathrm{S}(\mathrm{C}_p,\ \varepsilon)\cdot\cdot\varepsilon = \varepsilon\cdot\cdot\mathrm{M}(\mathrm{C}_p)\cdot\cdot\varepsilon > 0, \quad \forall\varepsilon \neq 0, \quad \mathrm{tr}\,\varepsilon = 0 \tag{3.3}$$

Since the incompressibility condition can be written in the form $\mathrm{tr}\,\mathrm{H} = 0$, the undetermined global tensor component $\sum$ does not take part in the inequality (2.10). Therefore, (2.10) is the convexity condition for the energy function in the Hencky tensor even in the case of an incompressible material /8/. Since the inequalities (2.9) and (2.10) are equivalent in this case, we arrive at the conclusion that the Hill inequality for an isotropic incompressible material will assure the uniqueness and stability of the p-configuration independently of the sign of the pressure p.

For an isotropic body the representation (3.1) has the form $\mathrm{S} = 2\mu\varepsilon + q\mathrm{E}$, and the inequality (3.3) reduces to the requirement $\mu > 0$.

Because of the incompressibility requirement (3.2), the vectors $\mathbf{a}, \mathbf{b}$ in the strengthened Hadamard inequality should be subjected to the condition $\mathbf{a}\cdot\mathbf{b} = 0$. For this reason, conditions (1.11) and (2.16) are equivalent for an isotropic incompressible material.

Since an isotropic incompressible material is not deformed in the p-configuration, and the Jaumann derivative of the stress tensor does not differ from the material derivative, the quantity μ is the shear modulus, i.e., the proportionality factor between the angle of small shear and the tangential stress due to this shear.

Let us consider the case of hydrostatic-pressure loading of an incompressible isotropic body, constrained by absolutely solid supports. Let us assume that the surface O bounding the body consists of three parts. The parts O_1 is fixed completely, the body makes contact with a smooth solid surface on the part O_2, and a uniform follower pressure p is applied on the surface O_3. Under such conditions the hydrostatic load will be conservative /9/.

The formulated problem has the evident solution: there are no displacements, and the stress tensor has the form $\mathrm{T} = -p\mathrm{E}$ at each point of the body. The body can hence be inhomogeneous.

The boundary conditions of the bifurcation problem for the mentioned equilibrium position have the form (1.9) on O_3 and $\mathbf{w} = 0$ on O_1. As is known /1/, the condition $\mathrm{N}\cdot\Theta = 0$ is satisfied in the absence of a surface load. Since the smooth solid surface produces no reaction in the tangent plane, it seems at first glance that the boundary conditions on the part O_2 of the surface will have the form

$$\mathrm{N}\cdot\mathbf{w} = 0, \quad \mathrm{N}\cdot\Theta\cdot\mathrm{D} = 0, \quad \mathrm{D} = \mathrm{E} - \mathbf{NN} \tag{3.4}$$

However, such a deduction will be erroneous in the general case. The correct boundary conditions can be obtained on O_2 by linearizing the condition

$$\mathrm{N}\cdot\mathrm{T}\cdot\mathrm{D} = 0 \tag{3.5}$$

From (3.5) we have

$$(\mathrm{N}\cdot\mathrm{T})^{\cdot}\cdot\mathrm{D} - \mathrm{N}\cdot\mathrm{T}\cdot(\mathrm{NN}^{\cdot} + \mathrm{N}^{\cdot}\mathrm{N}) = 0 \tag{3.6}$$

Since $\mathrm{N}^{\cdot} = \mathrm{D}\cdot\mathrm{N}^{\cdot}$, it follows from (3.5) that $\mathrm{N}\cdot\mathrm{T}\cdot\mathrm{N}^{\cdot} = 0$. Furthermore, we use the formulas obtained in /1/

$$(\mathrm{N}\cdot\mathrm{T})^{\cdot} = \mathrm{N}\cdot\Theta - \mathrm{N}\cdot\mathrm{T}\cdot\mathrm{N}\,(\nabla\cdot\mathbf{w}\mathrm{E} - \mathrm{N}\cdot\nabla\mathbf{w}\cdot\mathrm{N}), \quad \mathrm{N}^{\cdot} = \mathrm{N}\cdot(\nabla\cdot\mathbf{w}\mathrm{E} - \nabla\mathbf{w}^T) - \mathrm{N}\,(\nabla\cdot\mathbf{w} - \mathrm{N}\cdot\nabla\mathbf{w}\cdot\mathrm{N}) \tag{3.7}$$

Taking (3.7) into account, we obtain in place of (3.6)

$$\mathrm{N}\cdot\Theta\cdot\mathrm{D} + (\mathrm{N}\cdot\mathrm{T}\cdot\mathrm{N})\mathrm{N}\cdot(\nabla\mathbf{w}^T - \nabla\cdot\mathbf{w}\mathrm{E})\cdot\mathrm{D} = 0 \tag{3.8}$$

The incompressibility equation was not used in deriving the relationship (3.8), hence the linearized conditions of no friction on O_2 have the form (3.8) for any elastic body. The condition of impenetrability $\mathrm{N}\cdot\mathbf{w} = 0$ should still be appended. The expression in the left side of (3.8) can be converted to another form. The following identity is valid

$$\nabla\mathbf{w}^T - \nabla\cdot\mathbf{w}\mathrm{E} = \nabla\times(\mathrm{E}\times\mathbf{w}) \tag{3.9}$$

Let $x^\alpha\ (\alpha = 1, 2)$ be some Gaussian coordinates on the surface O_2, $\mathbf{R}_\beta$, $\mathbf{R}^\alpha$ the fundamental and reciprocal vector bases on O_2, and z the coordinate measured in a normal direction to the surface. We have the form (3.9)

$$\mathrm{N}\cdot(\nabla\mathbf{w}^T - \mathrm{E}\nabla\cdot\mathbf{w}) = \mathrm{N}\cdot(\mathrm{N}\times\mathrm{E}\times\partial\mathbf{w}/\partial z + \mathbf{R}^\beta\times\mathrm{E}\times\partial\mathbf{w}/\partial x^\beta) \tag{3.10}$$

We set that the derivative of the normal with respect to the direction drops out of the expression in the left side of (3.10), hence it is meaningful for a vector field $\mathbf{w}$ defined only on the surface O_2. We arrive at the relationship

$$\mathrm{N}\cdot(\nabla\mathbf{w}^T - \mathrm{E}\nabla\cdot\mathbf{w}) = \mathbf{R}^\alpha\mathrm{N}\cdot\partial\mathbf{w}/\partial x^\alpha - \mathrm{N}\mathbf{R}^\alpha\cdot\partial\mathbf{w}/\partial x^\alpha = \mathbf{R}^\alpha\,(\partial w/\partial x^\alpha + B_{\alpha\beta}u^\beta) - \mathrm{N}\mathbf{R}^\alpha\cdot\partial\mathbf{w}/\partial x^\alpha \tag{3.11}$$

$$w = \mathbf{w}\cdot\mathrm{N}, \quad u^\alpha = \mathbf{w}\cdot\mathbf{R}^\alpha, \quad B_{\alpha\beta} = -\mathbf{R}_\beta\cdot\partial\mathrm{N}/\partial x^\alpha$$

In another context, and without proof, formula (3.11) is presented in /10/. Taking into account (3.11) and the impenetrability condition, the relationship (3.8) acquires the form

$$\mathrm{N}\cdot\Theta\cdot\mathrm{D} + (\mathrm{N}\cdot\mathrm{T}\cdot\mathrm{N})B_{\alpha\beta}u^\beta\mathbf{R}^\alpha = 0 \tag{3.12}$$

It hence follows that the boundary conditions on O_2 can be written in the form (3.4) only when this surface is a part of a plane.

We obtain the final formulation of the boundary value problem about small deformations from a p-configuration for an isotropic incompressible body in the case of a follower pressure from (1.8) and (3.8):

$$\nabla\cdot\mathrm{S} = 0, \quad \nabla\cdot\mathbf{w} = 0, \quad \mathrm{S} = \mu\,(\nabla\mathbf{w} + \nabla\mathbf{w}^T) + q\mathrm{E}, \quad \mathbf{w} = 0 \text{ on } O_1; \quad \mathrm{N}\cdot\mathrm{S}\cdot\mathrm{D} = 0, \quad \mathrm{N}\cdot\mathbf{w} = 0 \text{ on } O_2 \tag{3.13}$$

$$\mathrm{N}\cdot\mathrm{S} = 0 \quad \text{on} \quad O_3 \tag{3.14}$$

Under the condition $\mu > 0$ the boundary value problem (3.13), (3.14) does not differ from the problem of *classical linear elasticity theory of an incompressible material*, and hence has just the trivial solution $\mathbf{w} = 0, q = 0$. This deduction is valid even for an inhomogeneous body, when the shear modulus is an arbitrary measurable function of the coordinates. The equality $\mu = 0$ is hence allowable in a set of measure zero.

Let us still consider this problem under the assumption that a pressure distributed uniformly over a surface O_3 is a dead load. In this case the boundary conditions on O_3 are formulated thus /1/:

$$\mathrm{N}\cdot\Theta = \mathrm{N}\cdot(\mathrm{S} + p\nabla\mathbf{w}^T) = 0 \tag{3.15}$$

We shall assume the body to be homogeneous $(\mu = \text{const} > 0)$, and the matrix of the coefficients of the second quadratic form $B_{\alpha\beta}$ positive-definite at every point of the surface O_2. Since $\mathbf{N}$ is the external normal relative to the body, it is seen from the last formula in (3.11) that the surface O_2 will be strictly concave in this case.

We will examine the functional $\psi(\mathbf{w})$ defined in a set of differentiable vectors $\mathbf{w}$ satisfying the incompressibility condition (3.2) and the conditions $\mathbf{w} = 0$ on O_1 and $\mathbf{N}\cdot\mathbf{w} = 0$ on O_2

$$\psi(\mathbf{w}) = \iiint_V \varepsilon\cdot\cdot\varepsilon dV \left[2\iiint_V \omega\cdot\omega dV + \iint_{O_2} B_{\alpha\beta}u^\alpha u^\beta dO\right]^{-1}, \quad \omega = {}^1/_2\nabla\times\mathbf{w} \tag{3.16}$$

It follows from the Korn inequalities /11/

$$\iiint_V \omega\cdot\omega dV \leqslant K_1 \iiint_V \varepsilon\cdot\cdot\varepsilon dV, \quad \iint_{O_2} B_{\alpha\beta}u^\alpha u^\beta dO \leqslant K_2 \iiint_V \varepsilon\cdot\cdot\varepsilon dV$$

that the functional $\psi(\mathbf{w})$ has a lower bound. Let ψ_0 be its exact lower bound. The following minimum variational principle holds.

If a function $\mathbf{w}_0$ exists such that $\psi(\mathbf{w}_0) = \psi_0$, then the boundary problem (3.13), (3.15) has the nontrivial solution $\mathbf{w} = \mathbf{w}_0$ for $p = p_0 = 2\mu\psi_0 / (1-\psi_0)$.

Proof. Since the function $\mathbf{w}_0$ makes the functional ψ a minimum, we have

$$\delta\psi \equiv \left[\frac{d}{d\gamma}\psi(\mathbf{w}_0 + \gamma\delta\mathbf{w})\right]_{\gamma=0} = 0 \tag{3.17}$$

We obtain from (3.16) and (3.17)

$$\iiint_V \varepsilon_0\cdot\cdot\delta\varepsilon dV = \psi_0\left[2\iiint_V \omega_0\cdot\delta\omega dV + \iint_{O_2} B_{\alpha\beta}u_0^\alpha \delta u^\beta dO\right] \tag{3.18}$$

The relationship (3.18) is the definition of the generalized solution of the problem under consideration of the bifurcation of equilibrium. Considering the function $\mathbf{w}_0$ twice continuously differentiable, we arrive at the formulation of the boundary value problem in the form (3.13), (3.15).

Indeed, (3.18) is converted as follows:

$$\iiint_V [(\varepsilon_0 + \psi_0 \mathbf{E}\times\omega_0)\cdot\cdot(\nabla\delta\mathbf{w})^T + q\nabla\cdot\delta\mathbf{w}]\, dV - \psi_0 \iint_{O_2} B_{\alpha\beta}u_0^\alpha\delta u^\beta dO = 0 \tag{3.19}$$

Here q is a functional Lagrange multiplier for the incompressibility condition. After application of the divergence theorem, we will have in place of (3.19)

$$-\iiint_V \nabla\cdot[\mu(\nabla\mathbf{w}_0 + \nabla\mathbf{w}_0^T) + q\mathbf{E} + p_0\nabla\mathbf{w}_0^T]\cdot\delta\mathbf{w}dV + \iint_O \mathbf{N}\cdot[\mu(\nabla\mathbf{w}_0 + \nabla\mathbf{w}_0^T) + q\mathbf{E} + p_0\nabla\mathbf{w}_0^T]\cdot\delta\mathbf{w}dO = p_0\iint_{O_2} B_{\alpha\beta}u_0^\alpha\delta u^\beta dO \tag{3.20}$$

Taking account of the support condition on O_1, the impenetrability condition on O_2, and the identity (3.11), we finally obtain from (3.20)

$$-\iiint_V \nabla\cdot[\mu(\nabla\mathbf{w}_0 + \nabla\mathbf{w}_0^T) + q\mathbf{E} + p_0\nabla\mathbf{w}_0^T]\cdot\delta\mathbf{w}dV + \iint_{O_2} \mathbf{N}\cdot[\mu(\nabla\mathbf{w}_0 + \nabla\mathbf{w}_0^T) + q\mathbf{E}]\cdot\mathbf{D}\cdot\delta\mathbf{w}dO + \tag{3.21}$$
$$\iint_{O_3} \mathbf{N}\cdot[\mu(\nabla\mathbf{w}_0 + \nabla\mathbf{w}_0^T) + q\mathbf{E} + p_0\nabla\mathbf{w}_0^T]\cdot\delta\mathbf{w}dO = 0$$

Hence (3.13), (3.15) results from (3.21) and the arbitrariness of the function $\delta\mathbf{w}$.

The author is grateful to L. P. Lebedev for useful comments.

REFERENCES

1. LUR'E, A. I., Theory of Elasticity. "Nauka", Moscow, 1970.

2. SEDOV, L. I., Introduction to the Mechanics of a Continuous Medium, Fizmatgiz, Moscow,1962.

3. LUR'E, A. I., On differentiation of tensors with respect to time. Trudy, Leningrad. Polytech. Inst., Vol.318, 1971.

4. FIKERA, G., Existence Theorems in Elasticity Theory, "Mir", Moscow, 1974.

5. ZUBOV, L. M., Variational principles of the nonlinear theory of elasticity. Case of superposition of a small deformation on a finite. PMM, Vol.35, No.5, 1971.

6. TRUESDELL, C., First course in the Rational Mechanics of a Continuous Media. "Mir", Moscow, 1975.

7. ZUBOV, L. M. On the Jaumann derivative for a second-rank tensor. Izv. Sev.- Kavkazsk. Nauchn. Tsentra Vyssh. Shkoly. Estestv. Nauki, No.2, 1976.

8. HILL, R. On constitutive inequalities for simple materials. J. Mech. and Phys. Solids, Vol.16, No.5, 1968.

9. ZUBOV, L. M., On conservation conditions of a hydrostatic load on a shell, Trudy, Tenth Conf. on Shell and Plate Theory, Vol.1, Kutaisi, 1975. "Metniereba", Tbilisi, 1975

10. ZUBOV, L. M., Theory of small strains of prestressed thin shells, PMM, Vol.40, No.1, 1976.

11. MOSOLOV, P. P. and MIASNIKOV, V. P., Proof of Korn inequality, Dokl. Akad. Nauk SSSR, Vol. 201, No.1, 1971.

Translated by M.D.F.